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On the quasi-Plücker coordinates

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Abstract

The quasideterminant defined by I. Gelfand and V. Retakh is a powerful tool for noncommutative theories. In this article, we review on the quasi-Plücker coordinates (a noncommutative analogue of the Plücker coordinates) with the quasideterminants and describe a proof of a fundamental case of left quasi-Plücker relations in a manner similar to the commutative one.

1 Introduction

First we recall the embedding of the Grassmannian $Gr_d(n)$ into $P^{\binom{n}{d}-1}$. The Plücker embedding is the map $\xi : Gr_d(n) \to P(\mathbf{R}^{\binom{n}{d}})$ defined as follows. For each d-subspace of $\mathbf{R}^n$, take the corresponding $n \times d$ matrix A and map it to the $\binom{n}{d}$-tuple of its maximal minors. We find that the map ξ is well-defined and injective. We call the coordinates of $p = [p_{1,\ldots,d} : \cdots : p_{n-d+1,\ldots,n}] \in P^{\binom{n}{d}-1}$ the Plücker coordinates.

Next, we consider the quasideterminant defined by I. Gelfand and V. Retakh. By using it, "noncommutative determinants" such as quaternionic determinants, superdeterminant, quantum determinant, Capelli determainant, etc. are expressed in the unified form [4]. In the theory of the noncommutative integrable system, quasideterminants are very useful to express the solution of the noncomutative integrable equations. [1], [6], [7], [8], [9], [10]. For example, in the noncommutative KdV equation

$$u_t = \frac{1}{4}(u_{xxx} + 3u_x u + 3u u_x),$$

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we put

$$y_k = e^{\xi(x,t,\alpha_k)} + a_k e^{-\xi(x,t,\alpha_k)},$$

$$b_i = (\partial_x W_i)W_i^{-1}, \quad W_i := |W(y_1, \cdots, y_i)|_{ii}$$

(W_i is (i,i)-**quasideterminant** of Wronskian matrix), then,

$$u(x,t) = 2\partial_x(\sum_{i=1}^{N} b_i)$$

are solutions of the noncommutative KdV equation (moreover, the noncommutative KdV hierarchy)[1].

Furthermore, various noncommutative analogue of theories using determinants are developed, for example, a noncommutative analogue of Cramer's formula, the Vandermonde determinant, symmetric functions, and so on. (see, [2], [3], [5] and references within).

In particular, we focus attention on the quasi-Plücker coordinates [2]. They are a noncommutative analogue of the Plücker coordinates. For example, if

$$A = \begin{pmatrix} a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \end{pmatrix},$$

left quasi-Plücker coordinates $q_{01}^{\{2\}}$, $q_{10}^{\{3\}}$, $q_{03}^{\{2\}}$, $q_{30}^{\{1\}}$ are

$$\begin{aligned} q_{01}^{\{2\}} &= (a_{10} - a_{12}a_{22}^{-1}a_{20})^{-1}(a_{11} - a_{12}a_{22}^{-1}a_{21}), \\ q_{10}^{\{3\}} &= (a_{11} - a_{13}a_{23}^{-1}a_{21})^{-1}(a_{10} - a_{13}a_{23}^{-1}a_{20}), \\ q_{03}^{\{2\}} &= (a_{10} - a_{12}a_{22}^{-1}a_{20})^{-1}(a_{13} - a_{12}a_{22}^{-1}a_{23}), \\ q_{30}^{\{1\}} &= (a_{13} - a_{11}a_{21}^{-1}a_{23})^{-1}(a_{10} - a_{11}a_{21}^{-1}a_{20}). \end{aligned}$$

They satisfy a noncommutative version of Plücker relations

$$q_{01}^{\{2\}} \cdot q_{10}^{\{3\}} + q_{03}^{\{2\}} \cdot q_{30}^{\{1\}} = 1.$$

In the commutative case, this identity implies a famous Plücker relation

$$p_{01} \cdot p_{23} - p_{02} \cdot p_{13} + p_{03} \cdot p_{12} = 0, \qquad \text{where} \quad p_{ij} = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}.$$

2 Quasideterminants

2.1 Definition

Let R be a (not necessary commutative) associative algebra. For a position (i, j) in a square matrix $A = (a_{rs})_{1 \leq r,s \leq n} \in M(n, R)$, let A^{ij} denote the $(n-1) \times (n-1)$-matrix obtained from A by deleting the i-th row and the j-th column. Let also $\mathrm{r}_i{}^j = (a_{i1}, \cdots, \hat{a}_{ij}, \cdots, a_{in})$ and $\mathrm{c}_j{}^i = (a_{1j}, \cdots, \hat{a}_{ij}, \cdots, a_{nj})^T$.

Definition 1. We assume that A^{ij} is invertible over R. The (i, j)-quasideterminant of A is defined by

$$|A|_{ij} = a_{ij} - \mathrm{r}_i{}^j \cdot (A^{ij})^{-1} \cdot \mathrm{c}_j{}^i. \tag{2.1}$$

Example 1. For $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$,

$$|A|_{11} = a_{11} - a_{12}a_{22}^{-1}a_{21}, \quad |A|_{12} = a_{12} - a_{11}a_{21}^{-1}a_{22},$$

$$|A|_{21} = a_{21} - a_{22}a_{12}^{-1}a_{11}, \quad |A|_{22} = a_{22} - a_{21}a_{11}^{-1}a_{12}.$$

It is sometimes convinient to adopt the following more explicit notation

$$|A|_{11} = \begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12}a_{22}^{-1}a_{21}.$$

Definition 2. For any (i, j), if all $|A|_{ij}^{-1}$ exist, then

$$A^{-1} = (|A|_{ji}^{-1})_{1 \leq i,j \leq n}.$$

Remark 2. *If the elements a_{ij} of the matrix A commute, then*

$$|A|_{ij} = (-1)^{i+j} \frac{\det A}{\det A^{ij}}.$$

By this definition 2, for fixed (i, j),

$$\begin{aligned} E &= AA^{-1} \\ \delta_{ik} &= \sum_{l=1}^{n} a_{il} |A|_{kl}^{-1} = a_{ij} |A|_{kj}^{-1} + \sum_{l \neq j} a_{il} |A|_{kl}^{-1} \end{aligned} \tag{2.2}$$

If $i \neq k$, we obtain the so-called **homological relations**. For example, in the case of $n = 2$, (2.2) is

$$\begin{aligned} -a_{ij} |A|_{kj}^{-1} &= a_{il} |A|_{kl}^{-1} \\ -|A|_{kj}\, a_{ij}^{-1} &= |A|_{kl}\, a_{il}^{-1} \\ -|A|_{ij}\, a_{sj}^{-1} &= |A|_{il}\, a_{sl}^{-1} \quad \text{for } A = \begin{pmatrix} a_{ij} & a_{il} \\ a_{sj} & a_{sl} \end{pmatrix}. \end{aligned}$$

In general, we have important identities as follows;

Proposition 3. *1. Row homological relations:*

$$-|A|_{ij} \cdot |A^{il}|_{sj}^{-1} = |A|_{il} \cdot |A^{ij}|_{sl}^{-1}, \qquad s \neq i$$

2. Column homological relations:

$$-|A^{kj}|_{it}^{-1} \cdot |A|_{ij} = |A^{ij}|_{kt}^{-1} \cdot |A|_{kj}, \qquad t \neq j$$

Remark 3. *By this proposition 3, $|A^{ij}|_{sl}^{-1}\, |A^{il}|_{sj}$ doesn't depend on s. Similarly, $|A^{kj}|_{it} |A^{ij}|_{kt}^{-1}$ doesn't depend on t. This property implies an invariance of the quasi-Plücker coordinates.*

Next, if $i = k$, (2.2) is

$$\begin{aligned}
1 &= a_{ij}|A|_{ij}^{-1} + \sum_{l \neq j} a_{il}|A|_{il}^{-1} \\
|A|_{ij} &= a_{ij} + \sum_{l \neq j} a_{il}|A|_{il}^{-1}|A|_{ij} \\
&= a_{ij} - \sum_{l \neq j} a_{il}|A^{ij}|_{sl}^{-1}|A^{il}|_{sj} \quad \text{(by row hom. rel.)}
\end{aligned}$$

Corollary 4 (Laplace expansions of quasideterminants).
For any $s \neq i$ and any $t \neq j$, then

$$\begin{aligned}
|A|_{ij} &= a_{ij} - \sum_{l \neq j} a_{il}|A^{ij}|_{sl}^{-1}|A^{il}|_{sj} \qquad \textit{(row expansion)}, \\
|A|_{ij} &= a_{ij} - \sum_{k \neq i} |A^{kj}|_{it}|A^{ij}|_{kt}^{-1}a_{kj} \qquad \textit{(column expansion)}
\end{aligned}$$

Example 4.

$$\begin{vmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} - a_{12}|A^{11}|_{s2}^{-1}|A^{12}|_{s1} - a_{13}|A^{11}|_{s3}^{-1}|A^{13}|_{s1}$$

$$= a_{11} - a_{12}\begin{vmatrix} \boxed{a_{22}} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{-1}\begin{vmatrix} \boxed{a_{21}} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} - a_{13}\begin{vmatrix} a_{22} & \boxed{a_{23}} \\ a_{32} & a_{33} \end{vmatrix}^{-1}\begin{vmatrix} \boxed{a_{21}} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11} - a_{12}\begin{vmatrix} a_{22} & a_{23} \\ \boxed{a_{32}} & a_{33} \end{vmatrix}^{-1}\begin{vmatrix} a_{21} & a_{23} \\ \boxed{a_{31}} & a_{33} \end{vmatrix} - a_{13}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & \boxed{a_{33}} \end{vmatrix}^{-1}\begin{vmatrix} a_{21} & a_{22} \\ \boxed{a_{31}} & a_{32} \end{vmatrix}.$$

Corollary 5. *Homological relations are rewritten as follows;*

- *Row homological relations:*

$$|A|_{ij} = |A|_{il} \cdot (-|A^{ij}|_{sl}^{-1}|A^{il}|_{sj}) = |A|_{il} \begin{vmatrix} & \vdots & & \vdots & \\ 0\cdots & \boxed{0} & \cdots 0 & 1 & 0\cdots \\ & \vdots & & \vdots & \\ & a_{sj} & & a_{sl} & \\ & \vdots & & \vdots & \end{vmatrix}_{ij}.$$

- *Column homological relations:*

$$|A|_{ij} = (-|A^{kj}|_{it}|A^{ij}|_{kt}^{-1}) \cdot |A|_{kj} = \begin{vmatrix} & 0 & & & \\ \cdots & \boxed{0} & \cdots & a_{it} & \cdots \\ & \vdots & & & \\ \cdots & 1 & \cdots & a_{kt} & \cdots \\ & \vdots & & & \end{vmatrix}_{ij} |A|_{kj}.$$

Proposition 6. *If A is a square matrix and i-th row of A is a left-linear combination of the other rows, then $|A|_{ij} = 0$.*
A column version of this is true as well.

Example 5.

$$\begin{aligned} & \begin{vmatrix} a_{11} & a_{12} & \boxed{a_{11}\lambda + a_{12}\mu} \\ a_{21} & a_{22} & a_{21}\lambda + a_{22}\mu \\ a_{31} & a_{32} & a_{31}\lambda + a_{32}\mu \end{vmatrix} \\ = \; & a_{11}\lambda + a_{12}\mu - (a_{11}\ a_{12}) \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}^{-1} \left\{ \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix} \lambda + \begin{pmatrix} a_{22} \\ a_{32} \end{pmatrix} \mu \right\} \\ = \; & a_{11}\lambda + a_{12}\mu - (a_{11}\ a_{12}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \lambda - (a_{11}\ a_{12}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mu \\ = \; & 0. \end{aligned}$$

2.2 Sylvester's Identity

Let $A = (a_{ij}) \in M(n, R)$ be a matrix and $A_0 = (a_{ij}), i, j = 1, \cdots, k$, a submatrix of A that is invertible over R. For $p, q = k+1, \cdots, n$, set

$$c_{pq} = \left| \begin{array}{ccc|c} & & & a_{1q} \\ & A_0 & & \vdots \\ & & & a_{kq} \\ \hline a_{p1} & \cdots & a_{pk} & \boxed{a_{pq}} \end{array} \right|. \tag{2.3}$$

Consider the $(n-k) \times (n-k)$ matrix

$$C = (c_{pq}), \quad p, q = k+1, \cdots, n.$$

The submatrix A_0 is called the *pivot* for the matrix C.

Theorem 7. *(Sylvester's identity) For $i, j = k+1, \cdots, n$,*

$$|A|_{ij} = |C|_{ij}.$$

Example 6.

$$\begin{vmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix}_{ij} \qquad (i, j = 2, 3)$$

Applying the theorem 7 with the $(1,1)$-entry 1 as a pivot, we put

$$c_{pq} = \begin{vmatrix} 1 & a_{1q} \\ 0 & a_{pq} \end{vmatrix}_{pq} = a_{pq} \quad (p, q = 2, 3)$$

and

$$\begin{vmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix}_{ij} = |C|_{ij} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}_{ij} \qquad (i, j = 2, 3).$$

3 Quasi-Plücker Coordinates

Definition 8. Let A be a $k \times n$ matrix $(k < n)$ and choose $i, j, i_1, \cdots, i_{k-1}$ such that $i \notin I = \{i_1, \cdots, i_{k-1}\}$. Define left quasi-Plücker coordinates $q_{ij}^I(A)$ of A by

$$q_{ij}^I(A) = \begin{vmatrix} a_{1i} & a_{1i_1} & \cdots & a_{1,i_{k-1}} \\ & & \cdots & \\ a_{ki} & a_{ki_1} & \cdots & a_{ki_{k-1}} \end{vmatrix}_{si}^{-1} \cdot \begin{vmatrix} a_{1j} & a_{1i_1} & \cdots & a_{1,i_{k-1}} \\ & & \cdots & \\ a_{kj} & a_{ki_1} & \cdots & a_{ki_{k-1}} \end{vmatrix}_{sj} \tag{3.1}$$

for an arbitrary s, $1 \leq s \leq k$. For example, if

$$A = \begin{pmatrix} a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \end{pmatrix}$$

and we put

$$(i\ j)_{si} \equiv \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}_{si} \qquad (s = 1 \text{ or } 2),$$

left quasi-Plücker coordinates $q_{01}^{\{2\}}$, $q_{10}^{\{3\}}$, $q_{03}^{\{2\}}$, $q_{30}^{\{1\}}$ of A are

$$q_{01}^{\{2\}} = (0\ 2)_{s0}^{-1}\ (1\ 2)_{s1},$$

$$q_{10}^{\{3\}} = (1\ 3)_{s1}^{-1}\ (0\ 3)_{s0},$$

$$q_{03}^{\{2\}} = (0\ 2)_{s0}^{-1}\ (3\ 2)_{s3},$$

$$q_{30}^{\{1\}} = (3\ 1)_{s3}^{-1}\ (0\ 1)_{s0}.$$

They satisfy the left quasi-Plücker relations

$$q_{01}^{\{2\}} \cdot q_{10}^{\{3\}} + q_{03}^{\{2\}} \cdot q_{30}^{\{1\}} = 1.$$

In the commutative case, this identity implies a Plücker relation

$$p_{01} \cdot p_{23} - p_{02} \cdot p_{13} + p_{03} \cdot p_{12} = 0, \qquad \text{where} \quad p_{ij} = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}.$$

"Right quasi-Plücker coodinates" are defined as well as left quasi-Plücker coodinates. For example, for

$$B = \begin{pmatrix} b_{01} & b_{02} \\ b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix},$$

the right quasi-Plücker coodinates of B are

$$r_{ij}^{\{k\}} = \begin{vmatrix} b_{i1} & b_{i2} \\ b_{k1} & b_{k2} \end{vmatrix}_{it} \begin{vmatrix} b_{j1} & b_{j2} \\ b_{k1} & b_{k2} \end{vmatrix}_{jt}^{-1},$$

and the right quasi-Plücker relation holds;

$$r_{01}^{\{3\}} r_{10}^{\{2\}} + r_{03}^{\{1\}} r_{30}^{\{2\}} = 1.$$

In the commutative case, this identity also implies a Plücker relation

$$p_{01} \cdot p_{23} - p_{02} \cdot p_{13} + p_{03} \cdot p_{12} = 0.$$

4 Quasi-Plücker Relations

Theorem 9 (left quasi-Plücker relations). *[2]*
Fix $M = (m_1, \cdots, m_{k-1})$, $L = (l_1, \cdots, l_k)$. *Let* $i \notin M$. *Then*

$$\sum_{j \in L} q_{ij}^{M} \cdot q_{ji}^{L \backslash \{j\}} = 1. \tag{4.1}$$

Here, we prove a left quasi-Plücker relation

$$q_{01}^{\{2\}} \cdot q_{10}^{\{3\}} + q_{03}^{\{2\}} \cdot q_{30}^{\{1\}} = 1 \tag{4.2}$$

in a manner similar to the commutative case. (On right quasi-Plücker relations, see [11].) First, we remark a Laplace expansion w.r.t. the 1st row of the quasideterminant

$$\begin{vmatrix} \boxed{0} & 0 & 0 & 1 \\ a_{20} & a_{21} & a_{23} & a_{22} \\ a_{10} & a_{11} & a_{13} & 0 \\ a_{20} & a_{21} & a_{23} & 0 \end{vmatrix} = - \begin{vmatrix} a_{21} & a_{23} & \boxed{a_{22}} \\ a_{11} & a_{13} & 0 \\ a_{21} & a_{23} & 0 \end{vmatrix}^{-1} \begin{vmatrix} \boxed{a_{20}} & a_{21} & a_{23} \\ a_{10} & a_{11} & a_{13} \\ a_{20} & a_{21} & a_{23} \end{vmatrix} = -a_{22}^{-1} \cdot 0 = 0.$$

Then, by using a homological relation, it is clear that the following quasideterminant is trivially zero;

$$\begin{vmatrix} \boxed{a_{10}} & a_{11} & a_{13} & a_{12} \\ a_{20} & a_{21} & a_{23} & a_{22} \\ a_{10} & a_{11} & a_{13} & 0 \\ a_{20} & a_{21} & a_{23} & 0 \end{vmatrix} \tag{4.3}$$

$$= \begin{vmatrix} a_{10} & a_{11} & a_{13} & \boxed{a_{12}} \\ a_{20} & a_{21} & a_{23} & a_{22} \\ a_{10} & a_{11} & a_{13} & 0 \\ a_{20} & a_{21} & a_{23} & 0 \end{vmatrix} \begin{vmatrix} \boxed{0} & 0 & 0 & 1 \\ a_{20} & a_{21} & a_{23} & a_{22} \\ a_{10} & a_{11} & a_{13} & 0 \\ a_{20} & a_{21} & a_{23} & 0 \end{vmatrix} = 0.$$

On the other hand, (4.3) is

$$
\begin{aligned}
&\begin{vmatrix} \boxed{a_{10}} & a_{11} & a_{13} & a_{12} \\ a_{20} & a_{21} & a_{23} & a_{22} \\ a_{10} & a_{11} & a_{13} & 0 \\ a_{20} & a_{21} & a_{23} & 0 \end{vmatrix} \\
= &\begin{vmatrix} \boxed{a_{10}} & a_{11} & a_{13} & a_{12} \\ a_{10} & a_{11} & a_{13} & 0 \\ a_{20} & a_{21} & a_{23} & 0 \\ a_{20} & a_{21} & a_{23} & a_{22} \end{vmatrix} \\
= &\begin{vmatrix} \boxed{\begin{vmatrix} \boxed{a_{10}} & a_{12} \\ a_{20} & a_{22} \end{vmatrix}} & \begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} \boxed{a_{13}} & a_{12} \\ a_{23} & a_{22} \end{vmatrix} \\ a_{10} & a_{11} & a_{13} \\ a_{20} & a_{21} & a_{23} \end{vmatrix}
\end{aligned}
$$

(Sylvester's identity with the $(4,4)$-entry a_{22} as a pivot)

$$
\begin{aligned}
= &\begin{vmatrix} \boxed{(0\ 2)_{10}} & (1\ 2)_{11} & (3\ 2)_{13} \\ a_{10} & a_{11} & a_{13} \\ a_{20} & a_{21} & a_{23} \end{vmatrix} \qquad \left((i\ j)_{si} = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}_{si} \right) \\
= &\ (0\ 2)_{10} - (1\ 2)_{11}\ (1\ 3)_{11}^{-1}\ (0\ 3)_{10} - (3\ 2)_{13}\ (1\ 3)_{13}^{-1}\ (0\ 1)_{10} \\
= &\ (0\ 2)_{10} - (1\ 2)_{11}\ (1\ 3)_{s1}^{-1}\ (0\ 3)_{s0} - (3\ 2)_{13}\ (3\ 1)_{s3}^{-1}\ (0\ 1)_{s0} \quad (s = 1 \text{ or } 2).
\end{aligned}
$$

This implies

$$
\begin{aligned}
1 &= (0\ 2)_{10}^{-1}\ (1\ 2)_{11}\ (1\ 3)_{s1}^{-1}\ (0\ 3)_{s0} + (0\ 2)_{10}^{-1}\ (3\ 2)_{13}\ (3\ 1)_{s3}^{-1}\ (0\ 1)_{s0} \\
&= (0\ 2)_{r0}^{-1}\ (1\ 2)_{r1}\ (1\ 3)_{s1}^{-1}\ (0\ 3)_{s0} + (0\ 2)_{r0}^{-1}\ (3\ 2)_{r3}\ (3\ 1)_{s3}^{-1}\ (0\ 1)_{s0} \\
&= q_{01}^{\{2\}} \cdot q_{10}^{\{3\}} + q_{03}^{\{2\}} \cdot q_{30}^{\{1\}} \qquad (r, s = 1 \text{ or } 2).
\end{aligned}
$$

Therefore, we have proved the left quasi-Plücker relation (4.2).

5 Comments on the Noncommutative KP Equation

In the commutative case, a bilinear form of KP equation is

$$(D_1^4 - 4D_1D_3 + 3D_2^2)\tau \cdot \tau = 0 \tag{5.1}$$

If τ is a Wronskian determinant, then (5.1) is nothing but a Plücker relation.

In the noncommutative case,

$$(v_t + v_{xxx} + 3v_x v_x)_x + 3v_{yy} - 3[v_x, v_y] = 0 \tag{5.2}$$

is called the noncommutative KP equation. In the commutative case, if we differentiate (5.2) wrt x and set $v_x = u$ to obtain the well-known (commuting) KP equation

$$(u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0.$$

Until now, no bilinearising transformation is found for (5.2).

Remark 7. *In [6], "quasiwronskian"*

$$Q(i,j) = \begin{vmatrix} \hat{\Theta} & e_{n-j} \\ \Theta^{(n+i)} & \boxed{0} \end{vmatrix}$$

(where $\hat{\Theta} = (\theta_k^{(l-1)})$ is a Wronskian matrix) and their derivatives are studied. In the commutative case,

$$Q(0,0) = -\frac{\tau_x}{\tau}, \quad \textit{where } \tau = |\hat{\Theta}|.$$

After some tedious calculations, they showed that

$$v = -2Q(0,0)$$

was a solution of the NC KP equation (5.2). However, a "τ-function" itself has been unclear.

6 Summary and Comments

- The quasi-Plücker coordinates satisfy the quasi-Plücker relations.
- There are other applications of the quasi-Plücker coordinates (Homological relations, noncommutative symmetric functions, noncommutative Gauss decomposition, etc.).
- We hope that the quasi-Plücker coordinates are applied to a theory for a noncommutative version of "τ-function" in the noncommutative integrable systems.

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